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The Wiener index of the zero-divisor graph of $\mathbb{Z}_n$

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ABSTRACT

The main objective of this article is to study the Wiener index of zero-divisor graph of the ring of integer modulo n. We present a constructed method to calculate the Wiener index of zero-divisor graph of $\mathbb{Z}_n$ for any positive integer n.

1. Introduction

Graph theory has provided chemists with a variety of useful tools, such as topological indices. In terms of theoretical chemistry it is known as a graph invariant which predicts the chemical properties of the molecule. Note that every molecule can be modified as a graph by representing atoms as vertices and chemical bonds as edges. There are two major classes of topological indices namely distance-based topological indices and degree-based topological indices of graphs. These classes of topological indices are widely applied in chemistry and pharmacology. The concept of topological index came from the work done by Wiener [25] while he was working on the boiling point of paraffin. He named this index as path number. Later on, the path number was renamed as the Wiener index. The Wiener index is mostly used to determine structure–property relationships. In particular, the Wiener index has a variety of applications in pharmaceutical science and in the structure of nanotubes. For results and applications of Wiener index, see [8–11,14].

On the other hand, the study of algebraic structures, using the properties of graph theory, tends to be an exciting research topic in the last two decades. The idea of the graph associated with zero-divisors of a commutative ring was introduced by Beck [6] in 1988. But, the present definition along with the name for the zero-divisor graph was first introduced by Anderson and Livingston in 1999. For recent results on zero-divisor graphs, readers may refer to [4,5,7,12,16,20,24]. For more details on zero-divisor graphs, one may refer to the survey article [3]. Note that the graphs constructed from the algebraic structure are highly symmetric and so they have some remarkable properties connecting chemical graph theory and networks in parallel computing. The graphs from the ring structure also found applications in molecular graphs and genetic code structure, refer [13].

The main goal of this paper is to find the Wiener index of the zero-divisor graph of the ring of integers modulo n. In Section 2, definitions and tools are developed for the calculation. In Section 3, the Wiener index of the zero-divisor graph of the ring of integers modulo n is determined for any positive integer n. It is expected that the investigation done here may have some interesting applications in molecular graphs, theoretical computer science and networking.

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2. Preliminaries

In this section, we summarize notations, concepts and results related to the Wiener index and zero-divisor graph which will be needed later.

Let G be a graph. The distance between two vertices x, y in G, denoted $d(x, y)$, is the number of edges on the shortest path between x and y. If there is no path connecting the two vertices, then the distance is defined as infinite. The *eccentricity* of the vertex u is the maximum distance from u to any vertex, that is, $e(u) = \max_{v \in V(G)} d(u, v)$. The *diameter* of a graph is the maximum eccentricity of any vertex in the graph and so $\text{diam}(G) = \max_{u \in V(G)} e(u)$.

The Wiener index is defined as the sum of the length of the shortest path between all pairs of vertices in the graph. In 1976, Hosoya [15] gave the mathematical representation for the Wiener index. The distance of a vertex u of a graph G, denoted by $d(u|G)$, is defined as,

$$d(u|G) = \sum_{v \in V(G)} d(u, v).$$

Then, the Wiener index of G is,

$$W(G) = \frac{1}{2} \sum_{u \in V(G)} d(u|G)$$

As mentioned, the present paper deals with what is known as the zero-divisor graph of a ring. The *zero-divisor graph* of a commutative ring R, denoted by $\Gamma(R)$, is a simple graph with all the non-zero zero-divisors as vertex set and distinct vertices $x, y \in R$ are adjacent if and only if $xy = 0$.

In recent years numerous works are done on topological indices of the graph. But to determine the value for the Wiener index for some graphs is a tedious job. So most of the works of the Wiener index were done for partial cases or done using algorithms. Let us review some of the work done on the topological indices of the zero-divisor graphs. Let p, q and r be distinct prime numbers. In 2011, Ahmadi et al. [1] provided an algorithm to determine the Wiener index of a zero-divisor graph of $\mathbb{Z}_n$ for $n = p^2, \; pq$. Later on in 2018, Mohammad et al. [18] has extended the result by determining the Wiener index of $\Gamma(\mathbb{Z}_n)$ for $n = p^m, p^m q$ where $m \in \mathbb{Z}, m \geq 2$ using the Hosoya polynomial. Recently in [24], the authors have determined the Wiener index of $\Gamma(\mathbb{Z}_n)$ for $n = p^m, pqr$ where $m \in \mathbb{N}$. In [12], the authors computed eccentric topological indices of zero-divisor graphs of $\mathbb{Z}_{pq} \times \mathbb{Z}_r$. The authors of [16], determined some edge-based eccentric topological indices of a zero-divisor graph of $\mathbb{Z}_{pq} \times \mathbb{Z}_{r^2}$. In [2], the first and second Zagreb indices of the zero-divisor graph were obtained from the ring $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$. Recently, Pirzada et al. [20] determined the Wiener index of a zero-divisor graph of $\mathbb{Z}_{p^m}$ for $m \in \mathbb{N}$. For the study on the Wiener index of other graphs from $\mathbb{Z}_n$, see [19,21,23]. It is worth mentioning that all the works on the topological indices of a zero-divisor graph are developed for some particular cases of the ring $\mathbb{Z}_n$. But, in this paper, we provide the mathematical formulation to determine the Wiener index of a zero-divisor graph of $\mathbb{Z}_n$ for any positive integer n.

Before moving into the main result, we require the following result, due to Lucas [17], who characterized all finite commutative ring according to the diameter value of $\Gamma(R)$.

Theorem 2.1 ([17, Theorem 2.6]). *Let R be a commutative ring. Then*

(1) *diam*$(\Gamma(R)) = 0$ *if and only if R is non reduced and isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[y]/(y^2)$.*
(2) *diam*$(\Gamma(R)) = 1$ *if and only if $xy = 0$ for each distinct pair of zero divisors and R has at least two nonzero zero divisors.*
(3) *diam*$(\Gamma(R)) = 2$ *if and only if either* (i) *R is reduced with exactly two minimal primes and at least three nonzero zero divisors, or* (ii) *$Z(R)$ is an ideal whose square is not (0) and each pair of distinct zero divisors has a nonzero annihilator.*
(4) *diam*$(\Gamma(R)) = 3$ *if and only if there are zero divisors $a \neq b$ such that $(0 : (a, b)) = (0)$ and either* (i) *R is a reduced ring with more than two minimal primes, or* (ii) *R is non reduced.*

Theorem 2.1 can be deduced to the ring $\mathbb{Z}_n$, as follows.

Proposition 2.2. *Let p_i be a prime number and $\alpha_i \in \mathbb{N}$ for $i = 1, \ldots, k$. Then the following statements hold true:*

(1) *diam*$(\Gamma(\mathbb{Z}_n)) = 0$ *if and only if either n is a prime or $n = 4$.*
(2) *diam*$(\Gamma(\mathbb{Z}_n)) = 1$ *if and only if n is a prime square.*
(3) *diam*$(\Gamma(\mathbb{Z}_n)) = 2$ *if and only if either* (i) *$n = p_1^{\alpha_1}$ with $\alpha_1 \geq 3$, or* (ii) *$n = p_1 \cdot p_2$.*
(4) *diam*$(\Gamma(\mathbb{Z}_n)) = 3$ *if and only if either* (i) *$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ with $3 \leq k \in \mathbb{N}$, or* (ii) *$n = p_1^{\alpha_1} p_2^{\alpha_2}$ with either $\alpha_1 \geq 2$ or $\alpha_2 \geq 2$.*

Now, we summarize the notations and results, which will be used to prove the main theorems.

Notations.

- For a positive integer n, the Euler's totient function is denoted by $\phi(n)$. If $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ where p_i's are distinct primes and $\alpha_i \in \mathbb{N}$ for all $i = 1, \ldots, k$, then

$$\phi(n) = \left(p_1^{\alpha_1} - p_1^{\alpha_1 - 1}\right) \cdots \left(p_k^{\alpha_k} - p_k^{\alpha_k - 1}\right).$$

- Let d be a proper divisor of n. Define the set $A_d = \{x \in \mathbb{Z}_n : gcd(x, n) = d\}$.
- For a vertex u in G, the notation $N_i(u) = \{v \in V(G) : d(u, v) = i\}$ for $1 \leq i \leq diam(\Gamma(\mathbb{Z}_n))$.

Remark 2.3. Let $d_1, d_2, \ldots, d_\ell$ be the distinct proper divisors of n and let $A_{d_j} = \{x \in \mathbb{Z}_n : gcd(x, n) = d_j\}$ for $j = 1, \ldots, \ell$. Then the following statements hold true.

(1) The sets A_{d_j} for $1 \leq j \leq \ell$ are pairwise disjoint and $V(\Gamma(\mathbb{Z}_n)) = \bigcup_{j=1}^{\ell} A_{d_j}$.
(2) [26, Proposition 2.1] $|A_{d_j}| = \phi(\frac{n}{d_j})$ for $1 \leq j \leq \ell$.
(3) [7, Lemma 2.4] For $j, k \in 1, 2, \ldots, \ell$, a vertex of A_{d_j} is adjacent to a vertex of A_{d_k} in $\Gamma(\mathbb{Z}_n)$ if and only if n divides $d_j \cdot d_k$.
(4) Denote the notation $A_{\langle n/d_j \rangle} = \bigcup_{d_k \in \langle n/d_j \rangle} A_{d_k}$ where d_k's are divisors of n and $\langle n/d_j \rangle = \{n/d_j, 2n/d_j, \ldots, (d_j - 1)n/d_j\}$. So, by part (3), the neighborhood of each vertex in A_{d_j} is $A_{\langle n/d_j \rangle}$ in $\Gamma(\mathbb{Z}_n)$.

We close this section by finding the degree of a vertex in the zero-divisor graph of $\mathbb{Z}_n$ according to our notations.

Proposition 2.4. *Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ where p_i's are distinct primes and $k, \alpha_i \in \mathbb{N}$. Let $d = p_1^{\beta_1} \cdots p_k^{\beta_k}$ be a proper divisor of n and $x \in A_d$. Then, in $\Gamma(\mathbb{Z}_n)$,*

$$deg(x) = \begin{cases} d - 2 & \text{if } \beta_i \geq \lceil \alpha_i/2 \rceil \text{ for all } i = 1, \ldots, k \\ d - 1 & \text{otherwise.} \end{cases}$$

Proof. Let $x \in A_d$. In $\Gamma(\mathbb{Z}_n)$, any vertex x is adjacent to all the vertices of $Ann(x)^* = \{y \in \mathbb{Z}_n^* : x \cdot y = 0\}$. Note that $|Ann(x)^*| = gcd(x, n) - 1 = d - 1$. Further $x \in Ann(x)^*$ if and only if $\beta_i \geq \lceil \alpha_i/2 \rceil$ for all $i = 1, \ldots, k$. Since there is no loop for any vertex of the graph $\Gamma(\mathbb{Z}_n)$, we get the required claim. □

3. Main results

In this section, we have explicitly given a formula for determining the Wiener index of a zero-divisor graph of $\mathbb{Z}_n$ for any $n \in \mathbb{N}$. The corresponding results are given in Theorems 3.1, 3.4 and 3.8.

The first theorem of this kind deals with the zero-divisor graph of $\mathbb{Z}_n$ when n is a prime power. For result regarding the Wiener index of a zero-divisor graph of this case, refer [20, Theorem 8]. It is worthwhile to note that the calculation part of the formula for $W(\Gamma(\mathbb{Z}_n))$ given in the following result is much simpler than the one given in Theorem 8 [20]. Further, the second part of Theorem 3.1 is mentioned in Corollary 2.11 of [18].

Theorem 3.1. *Let p be prime number and $\alpha \in \mathbb{N}$. Then*

(i) $W(\Gamma(\mathbb{Z}_p)) = W(\Gamma(\mathbb{Z}_4)) = 0$.
(ii) $W(\Gamma(\mathbb{Z}_{p^\alpha})) = \frac{1}{2}\left[2p^{2(\alpha-1)} - (\alpha - 1)p^\alpha + (\alpha - 6)p^{\alpha-1} + p^{\alpha - \lceil \frac{\alpha}{2} \rceil} + 2\right]$ *where $\alpha \geq 2$ and $p^\alpha \neq 4$.*

Proof. Let $n \geq 2$ be a positive integer.

(i) Clearly $Z(\mathbb{Z}_p)^* = \emptyset$ and $Z(\mathbb{Z}_4)^*$ is a singleton set so that $W(\Gamma(\mathbb{Z}_p)) = W(\Gamma(\mathbb{Z}_4)) = 0$.

(ii) Suppose $n = p^\alpha$ for some prime p and $2 \leq \alpha \in \mathbb{N}$. If $\alpha = 2$, then $|Z(\mathbb{Z}_n)^*| = p - 1$ and $\Gamma(\mathbb{Z}_n) = K_{p-1}$. Therefore $W(\Gamma(\mathbb{Z}_n)) = \frac{(p-1)(p-2)}{2}$.

Let $\alpha \geq 2$. In this case, any proper divisor of n is of the form p^j for some $j \in \{1, \ldots, \alpha - 1\}$. Let $d_j = p^j$ for all $j \in \{1, \ldots, \alpha - 1\}$ and let $x \in A_{d_j}$. Then, by Proposition 2.4,

$$deg(x) = \begin{cases} p^j - 1 & \text{if } j < \lceil \frac{\alpha}{2} \rceil \\ p^j - 2 & \text{if } j \geq \lceil \frac{\alpha}{2} \rceil. \end{cases}$$

Now, by Proposition 2.2, we have $diam(\Gamma(\mathbb{Z}_{p^\alpha})) = 2$. Since $|V(\Gamma(\mathbb{Z}_{p^\alpha}))| = p^{\alpha-1} - 1$, we get

$$d(x|G) = \begin{cases} (p^j - 1) + 2(p^{\alpha-1} - p^j - 1) & \text{if } j < \lceil \frac{\alpha}{2} \rceil \\ (p^j - 2) + 2(p^{\alpha-1} - p^j) & \text{if } j \geq \lceil \frac{\alpha}{2} \rceil. \end{cases}$$

Note that $|A_{d_j}| = \phi(\frac{n}{d_j}) = p^{\alpha-j} - p^{\alpha-j-1}$ for $1 \le j \le \alpha - 1$. Therefore,

$$\begin{aligned}
W(\Gamma(\mathbb{Z}_{p^\alpha})) &= \frac{1}{2}\sum_{j=1}^{\lceil\frac{\alpha}{2}\rceil-1}(p^{\alpha-j}-p^{\alpha-j-1})(p^j-1) + \sum_{j=1}^{\lceil\frac{\alpha}{2}\rceil-1}(p^{\alpha-j}-p^{\alpha-j-1})(p^{\alpha-1}-p^j-1)\\
&\quad + \frac{1}{2}\sum_{j=\lceil\frac{\alpha}{2}\rceil}^{\alpha-1}(p^{\alpha-j}-p^{\alpha-j-1})(p^j-2) + \sum_{j=\lceil\frac{\alpha}{2}\rceil}^{\alpha-1}(p^{\alpha-j}-p^{\alpha-j-1})(p^{\alpha-1}-p^j)\\
&= \frac{1}{2}\left[\left\lceil\frac{\alpha}{2}\right\rceil p^\alpha - p^\alpha - \left\lceil\frac{\alpha}{2}\right\rceil p^{\alpha-1} + p^{\alpha-\lceil\frac{\alpha}{2}\rceil}\right]\\
&\quad + p^\alpha - \left\lceil\frac{\alpha}{2}\right\rceil p^\alpha - 2p^{\alpha-1} + \left\lceil\frac{\alpha}{2}\right\rceil p^{\alpha-1} + p^{2\alpha-2} - p^{2\alpha-\lceil\frac{\alpha}{2}\rceil-1} + p^{\alpha-\lceil\frac{\alpha}{2}\rceil}\\
&\quad + \frac{1}{2}[\alpha p^\alpha - \alpha p^{\alpha-1} - \left\lceil\frac{\alpha}{2}\right\rceil p^\alpha + \left\lceil\frac{\alpha}{2}\right\rceil p^{\alpha-1} - 2p^{\alpha-\lceil\frac{\alpha}{2}\rceil} + 2]\\
&\quad + \alpha p^{\alpha-1} - \alpha p^\alpha - \left\lceil\frac{\alpha}{2}\right\rceil p^{\alpha-1} + \left\lceil\frac{\alpha}{2}\right\rceil p^\alpha - p^{\alpha-1} + p^{2\alpha-\lceil\frac{\alpha}{2}\rceil-1}\\
&= \frac{1}{2}\left[2p^{2(\alpha-1)} - (\alpha-1)p^\alpha + (\alpha-6)p^{\alpha-1} + p^{\alpha-\lceil\frac{\alpha}{2}\rceil} + 2\right]. \quad \square
\end{aligned}$$

Using Theorem 3.1, one can obtain the following corollary which was proved by Reddy et al. in [22] and Mohammad et al. in [18].

Corollary 3.2. *Let p be prime number. Then*

(i) $W(\Gamma(\mathbb{Z}_{p^2})) = \frac{(p-1)(p-2)}{2}$.
(ii) *[18, Corollary 2.11], [22, Theorem 5.1]* $W(\Gamma(\mathbb{Z}_{p^3})) = \frac{1}{2}\left[2p^4 - 2p^3 - 3p^2 + p + 2\right] = \left(\frac{p-1}{2}\right)\left(2p^3 - 3p - 2\right)$.

In order to prove the second theorem, we need a lemma, which we state and prove below.

Lemma 3.3. *Suppose $n = p_1 \cdots p_k$ where p_i's are distinct primes and $2 \le k \in \mathbb{N}$. Let $d = p_1^{\beta_1} \cdots p_k^{\beta_k}$ be a proper divisor of n and $x \in A_d$. Then, in $\Gamma(\mathbb{Z}_n)$, $e(x) = 2$ if and only if $d = p_1 \cdots p_{r-1} \cdot p_{r+1} \cdots p_k$ where $1 \le r \le k$.*

Proof. Since d is a proper divisor of n, we have $\beta_i \in \{0, 1\}$ for all $i = 1, \dots, k$ together with $d \neq 1$ and $d \neq n$.

($\Leftarrow$): Assume that there exists a unique $r \in \{1, \dots, k\}$ such that $\beta_r = 0$. Let $y \in Z(\mathbb{Z}_n)^* \setminus A_{\langle\frac{n}{d}\rangle}$. Implies that y is of the form $p_1^{\gamma_1} \cdots p_{r-1}^{\gamma_{r-1}} \cdot p_{r+1}^{\gamma_{r+1}} \cdots p_k^{\gamma_k}$ where $\gamma_s \in \{0, 1\}$ for all $s = 1, \dots, r-1, r+1, \dots, k$. Then $z = p_1^{1-\gamma_1} \cdots p_{r-1}^{1-\gamma_{r-1}} \cdot p_r \cdot p_{r+1}^{1-\gamma_{r+1}} \cdots p_k^{1-\gamma_k} \in A_{\langle\frac{n}{d}\rangle}$ so that $x - z - y$ is a path in $\Gamma(\mathbb{Z}_n)$ and therefore $d(x, y) = 2$.

($\Rightarrow$): Assume that $e(x) = 2$. Suppose, on the contrary, that there exist two distinct $t', t'' \in \{1, \dots, k\}$ such that $\beta_{t'} = \beta_{t''} = 0$. Choose $y = p_{t'} \in Z(\mathbb{Z}_n)^* \setminus A_{\langle\frac{n}{d}\rangle}$. The fact that the vertices of $\Gamma(\mathbb{Z}_n)$ are the proper divisors of n implies that, for every $z = p_1^{\gamma_1} \cdots p_k^{\gamma_k} \in A_{\langle\frac{n}{d}\rangle}$, $\gamma_s = 0$ for some $s \in \{1, \dots, k\} \setminus \{t', t''\}$. Therefore p_s is not available on the prime divisors of $y \cdot z$ and so y is not adjacent to any vertices in $A_{\langle\frac{n}{d}\rangle}$. Thus $d(x, y) > 2$, a contradiction. $\square$

We are now in the position to state and prove the second theorem which determines the formula for $W(\Gamma(\mathbb{Z}_n))$ when $n = p_1 \cdots p_k$ for $2 \le k \in \mathbb{N}$.

Theorem 3.4. *Let $n = p_1 \cdots p_k$ where p_i's are distinct primes and $2 \le k \in \mathbb{N}$. Let d_j be a proper divisor of n for $j = 1, \dots, 2^k - 2$.*

- *For $1 \le j \le k$, let $d_j = p_1 \cdots p_{j-1} \cdot p_{j+1} \cdots p_k$;*
- *For $k + 1 \le j \le 2^k - 2$, let $d_j = p_1^{\beta_1} \cdots p_k^{\beta_k}$ where $\beta_i = 0 = \beta_{i'}$ for some distinct $i, i' \in \{1, \dots, k\}$. In this case, let $Z_j = \{i \in \{1, \dots, k\} : \beta_i = 0\}$ and let $Z_j = \{i_1, \dots, i_{r(j)}\}$. For $1 \le \ell \le 2^{r(j)} - 2$, define $\tau_{\ell(j)} = p_{i_1}^{\gamma_{i_1}} \cdots p_{i_{r(j)}}^{\gamma_{i_{r(j)}}}$ where $\gamma_{i_s} \in \{0, 1\}$ for all $1 \le s \le r(j)$ with $\gamma_{i_t} = 0$ for some $1 \le t \le r(j)$.*

Then

$$W(\Gamma(\mathbb{Z}_n)) = \frac{1}{2}\left[\sum_{j=1}^{2^k-2}\left(\phi(\frac{n}{d_j}) \cdot (2(n - \phi(n)) - 3 - d_j)\right)\right] + \frac{1}{2}\left[\sum_{j=k+1}^{2^k-2}\left(\phi(\frac{n}{d_j}) \cdot \sum_{\ell(j)=1}^{2^{r(j)}-2}\phi(\frac{n}{\tau_{\ell(j)}})\right)\right].$$

Proof. Let $n = p_1 \cdots p_k$ where $k \ge 2$. Then the number of proper divisors of n is $2^k - 2$. For $1 \le j \le 2^k - 2$, let us consider an arbitrary proper divisor of n as $d_j = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdots p_k^{\beta_k}$ where $\beta_i \in \{0, 1\}$. Let $x \in A_{d_j}$. Since $d_j \neq n$, we have $\beta_\ell = 0$ for some $\ell \in \{1, \ldots, k\}$. So, by Proposition 2.4,

$$|N_1(x)| = deg(x) = d_j - 1. \tag{1}$$

The following facts are followed from Lemma 3.3;

- $e(x) = 2$ if and only if there exists a unique $r \in \{1, \ldots, k\}$ such that $\beta_r = 0$.
- $e(x) = 3$ if and only if there exist $r', r'' \in \{1, \ldots, k\}$ such that $\beta_{r'} = \beta_{r''} = 0$, because $diam(\Gamma(\mathbb{Z}_n)) = 3$.
- $e(x) = e(y)$ for all $y \in A_{d_j}$.

If unique $r \in \{1, \ldots, k\}$ such that $\beta_r = 0$, then by Eq. (1),

$$|N_2(x)| = |Z(\mathbb{Z}_n)^* \setminus \{x\}| - deg(x) = n - \phi(n) - d_j - 1.$$

Therefore,

$$\begin{aligned} d(x|G) &= (d_j - 1) + 2(n - \phi(n) - d_j - 1) \\ &= 2\,(n - \phi(n)) - d_j - 3. \end{aligned} \tag{2}$$

Let $\beta_{r'} = \beta_{r''} = 0$ for some $r', r'' \in \{1, \ldots, k\}$. Now, rearrange p_i's such that $\beta_i = 0$ for $1 \le i \le r$ and $\beta_i = 1$ for $r + 1 \le i \le k$. That is $d_j = p_{r+1} \cdots p_k$. Clearly, in this case, $r \ge 2$.

Let $y \in A_{d_{j'}}$ and $d_{j'} = p_1^{\gamma_1} \cdots p_k^{\gamma_k}$ where $\gamma_i \in \{0, 1\}$ for all $i = 1, \ldots, k$.

Claim. *$d(x, y) = 2$ if and only if $\gamma_\ell = 0$ for some $\ell \in \{1, \ldots, r\}$ and $\gamma_{\ell'} = 1$ for some $\ell' \in \{r + 1, \ldots, k\}$.*

($\Rightarrow$): Assume that $d(x, y) = 2$. Suppose, on the contrary, that either $d_{j'} = p_1 \cdots p_r \cdot p_{r+1}^{\gamma_{r+1}} \cdots p_k^{\gamma_k}$ or $d_{j'} = p_1^{\gamma_1} \cdots p_r^{\gamma_r}$ with $\gamma_t = 0$ for some $t \in \{1, \ldots, r\}$. If $d_{j'} = p_1 \cdots p_r \cdot p_{r+1}^{\gamma_{r+1}} \cdots p_k^{\gamma_k}$, then x is adjacent to y in $\Gamma(\mathbb{Z}_n)$, a contradiction. If $d_{j'} = p_1^{\gamma_1} \cdots p_r^{\gamma_r}$ with $\gamma_t = 0$ for some $1 \le t \le r$, then, in $\Gamma(\mathbb{Z}_n)$, y is adjacent to all the vertices z of the form $p_1^{\lambda_1} \cdots p_r^{\lambda_r} \cdot p_{r+1} \cdots p_k$ with $1 - \gamma_i \le \lambda_i \le 1$ for all $i \in \{1, \ldots, r\}$. Note that $\lambda_t = 1$. Implies that $\lambda_{t'} = 0$ for some $t' \in \{1, \ldots, t - 1, t + 1, \ldots, r\}$ because z is a proper divisor of n. Since $\beta_{t'} = 0$, z is not adjacent to x so that $d(x, y) > 2$, a contradiction.

($\Leftarrow$): Assume that $d_{j'} = p_1^{\gamma_1} \cdots p_k^{\gamma_k}$ with $\gamma_\ell = 0$ for some $\ell \in \{1, \ldots, r\}$ and $\gamma_{\ell'} = 1$ for some $\ell' \in \{r+1, \ldots, k\}$. So x is not adjacent to y. Choose $z = p_1 \cdots p_r \cdot p_{r+1}^{1-\gamma_{r+1}} \cdots p_k^{1-\gamma_k}$. Since $\gamma_{\ell'} = 1$, we have $z \in V(\Gamma(\mathbb{Z}_n))$. Clearly $x - z - y$ is a path in $\Gamma(\mathbb{Z}_n)$ so that $d(x, y) = 2$. Therefore, the claim holds true.

For $x \in A_{d_j}$ and $1 \le m \le 3$, let us denote σ_{jm} as the number of proper divisors d of n such that $d(x, y) = m$ for $y \in A_d$. So, to find $|N_2(x)|$ and $|N_3(x)|$, we have to calculate σ_{j2} and σ_{j3}. Note that, in this case, $d_j = p_{r+1} \cdots p_k$. So x is adjacent to all the vertices of the sets A_d with d of the form $p_1 \cdots p_r \cdot p_{r+1}^{\gamma_{r+1}} \cdots p_k^{\gamma_k}$ where $\gamma_i \in \{0, 1\}$ for $i \in \{r + 1, \ldots, k\}$. Therefore, $\sigma_{j1} = 2^{k-r} - 1$. Also, by the claim, the sets A_d's of distance two from x are of the form $d = p_1^{\gamma_1} \cdots p_k^{\gamma_k}$ with $\gamma_\ell = 0$ for at least one $\ell \in \{1, \ldots, r\}$ and $\gamma_{\ell'} = 1$ for at least one $\ell' \in \{r + 1, \ldots, k\}$. So, $\sigma_{j2} = (2^r - 1) \cdot \sum_{\ell=1}^{k-r} 2^{k-r-\ell} = (2^r - 1) \cdot \left(2^{k-r} - 1\right)$. Further, $d(x, y) = 3$ when d is of the form $p_1^{\gamma_1} \cdots p_r^{\gamma_r}$ with $\gamma_t = 0$ for at least one $t \in \{1, \ldots, r\}$. Therefore, $\sigma_{j3} = \sum_{\ell=1}^{r-1} 2^{r-\ell} = 2^r - 2$.

Here, notice that $\sigma_{j3} \le \sigma_{j2}$. So to reduce the number of terms to be calculated for finding $W(\Gamma(\mathbb{Z}_n))$, we use σ_{j3} in place of σ_{j2} by subtracting it from $|V(\Gamma(\mathbb{Z}_n))|$. Note that, if $x \in A_{d_j}$ where $d_j = p_{r+1} \cdots p_k$, then the sets A_{d_m} corresponding to σ_{j3} is $d_m = p_1^{\gamma_1} \cdots p_r^{\gamma_r}$ with $\gamma_t = 0$ for some $t \in \{1, \ldots, r\}$. Let us denote these d_m's by $\tau_1, \ldots, \tau_{\sigma_{j3}}$. Hence,

$$|N_3(x)| = \sum_{s=1}^{\sigma_{j3}} \phi(\frac{n}{\tau_s}) \tag{3}$$

so that

$$|N_2(x)| = n - \phi(n) - d_j - \sum_{s=1}^{\sigma_{j3}} \phi(\frac{n}{\tau_s}) - 1. \tag{4}$$

Therefore, by Eqs. (3) and (4), in case of $e(x) = 3$, we have

$$\begin{aligned} d(x|G) &= (d_j - 1) + 3\left(\sum_{s=1}^{\sigma_{j3}} \phi(\frac{n}{\tau_s})\right) + 2\left(n - \phi(n) - d_j - \sum_{s=1}^{\sigma_{j3}} \phi(\frac{n}{\tau_s}) - 1\right) \\ &= 2\,(n - \phi(n)) - d_j + \sum_{s=1}^{\sigma_{j3}} \phi(\frac{n}{\tau_s}) - 3. \end{aligned} \tag{5}$$

Finally, to find the Wiener index of $\Gamma(\mathbb{Z}_n)$, we have to find the number of choices for j's such that $e(x) = 2$ for $x \in A_{d_j}$. Clearly, by Lemma 3.3, there are k such choices available for j's. For $1 \le j \le k$, let $d_j = p_1 \cdots p_{j-1} \cdot p_{j+1} \cdots p_k$.

Thus, by Eqs. (2) and (5),

$$\begin{aligned}
W(\Gamma(\mathbb{Z}_n)) &= \frac{1}{2}\left[\sum_{j=1}^{k}\left(\phi(\frac{n}{d_j})\cdot d(x|G)_{e(x)=2}\right)\right] + \frac{1}{2}\left[\sum_{j=k+1}^{2^k-2}\left(\phi(\frac{n}{d_j})\cdot d(x|G)_{e(x)=3}\right)\right] \\
&= \frac{1}{2}\left[\sum_{j=1}^{k}\left(\phi(\frac{n}{d_j})\cdot(2(n-\phi(n))-d_j-3)\right)\right] \\
&\quad + \frac{1}{2}\left[\sum_{j=k+1}^{2^k-2}\left(\phi(\frac{n}{d_j})\cdot\left(2\,(n-\phi(n))-d_j+\sum_{s=1}^{\sigma_{j3}}\phi(\frac{n}{\tau_s})-3\right)\right)\right] \\
&= \frac{1}{2}\left[\sum_{j=1}^{2^k-2}\left(\phi(\frac{n}{d_j})\cdot(2(n-\phi(n))-d_j-3)\right)\right] \\
&\quad + \frac{1}{2}\left[\sum_{j=k+1}^{2^k-2}\left(\phi(\frac{n}{d_j})\cdot\sum_{s=1}^{\sigma_{j3}}\phi(\frac{n}{\tau_s})\right)\right]. \quad \square
\end{aligned}$$

Now we restrict the attention to n as the product of either two or three primes and explicatively given the value of corresponding Wiener index of $\Gamma(\mathbb{Z}_n)$ in terms of its prime decomposition.

Corollary 3.5. *Let p_1, p_2 and p_3 be distinct primes. Then*
(i) $W(\Gamma(\mathbb{Z}_{p_1p_2})) = p_1^2 + p_2^2 + p_1p_2 - 4p_1 - 4p_2 + 5.$

$$\begin{aligned}
\text{(ii)}\,W(\Gamma(\mathbb{Z}_{p_1p_2p_3})) = &\,3(p_1^2p_2p_3 + p_1p_2^2p_3 + p_1p_2p_3^2) - 15(p_1p_2p_3) + (p_1^2p_2^2 + p_2^2p_3^2 + p_1^2p_3^2) \\
&- 3\left[p_1^2(p_2+p_3) + p_2^2(p_1+p_3) + p_3^2(p_1+p_2)\right] \\
&+ 8(p_1p_2 + p_2p_3 + p_1p_3) + 2(p_1^2+p_2^2+p_3^2) - 4(p_1+p_2+p_3) + 3.
\end{aligned}$$

We now illustrate Theorem 3.4 when $n = 210$.

Remark 3.6. Consider $n = 2\cdot3\cdot5\cdot7 = 210$. Then the number of proper divisors of n is $2^4 - 2 = 14$. Here $d_1 = 3\cdot5\cdot7$, $d_2 = 2\cdot5\cdot7$, $d_3 = 2\cdot3\cdot7$ and $d_4 = 2\cdot3\cdot5$. Let $d_5 = 2\cdot3$, $d_6 = 2\cdot5$, $d_7 = 2\cdot7$, $d_8 = 3\cdot5$, $d_9 = 3\cdot7$, $d_{10} = 5\cdot7$, $d_{11} = 2$, $d_{12} = 3$, $d_{13} = 5$ and $d_{14} = 7$.

For instance, we illustrate in detail for d_6 and d_{11}. In general, let $d_j = 2^{\beta_1}\cdot3^{\beta_2}\cdot5^{\beta_3}\cdot7^{\beta_4}$ where $\beta_i \in \{0,1\}$ for $j = 1,\dots,14$.

Consider $d_6 = 2\cdot5$. Here $\beta_2 = 0$ and $\beta_4 = 0$. Implies that $Z_6 = \{2,4\}$ and so $r(6) = 2$. Consequently $\tau_{1(6)} = 3$ and $\tau_{2(6)} = 7$. Therefore $\sum_{\ell(6)=1}^{2}\phi(\frac{n}{\tau_{\ell(6)}}) = 24 + 8 = 32$.

For $d_{11} = 2$, we have $Z_{11} = \{2,3,4\}$ and so $r(11) = 3$. Consequently $\tau_{1(11)} = 3$, $\tau_{2(11)} = 5$, $\tau_{3(11)} = 7$, $\tau_{4(11)} = 3\cdot5$, $\tau_{5(11)} = 3\cdot7$ and $\tau_{6(11)} = 5\cdot7$. Therefore $\sum_{\ell(11)=1}^{6}\phi(\frac{n}{\tau_{\ell(11)}}) = 24 + 12 + 8 + 6 + 4 + 2 = 56$.

Note that $n - \phi(n) = 210 - 48 = 160$.

$$\begin{aligned}
W(\Gamma(\mathbb{Z}_{210})) &= \frac{1}{2}\left[\sum_{j=1}^{14}\left(\phi(\frac{n}{d_j})\cdot(2\times160-3-d_j)\right)\right] + \frac{1}{2}\left[\sum_{j=5}^{14}\left(\phi(\frac{n}{d_j})\cdot\sum_{\ell(j)=1}^{2^{r(j)}-2}\phi(\frac{n}{\tau_{\ell(j)}})\right)\right] \\
&= \frac{1}{2}[50184 + 9120] = 29652.
\end{aligned}$$

In order to prove the third theorem, we need a lemma, which we state and prove below.

Lemma 3.7. *Let $n = p_1\cdots p_w\cdot p_{w+1}^{\alpha_{w+1}}\cdots p_k^{\alpha_k}$ where p_i's are distinct primes, $w \in \mathbb{W}$, $2 \le \alpha_i \in \mathbb{N}$ and $2 \le k \in \mathbb{N}$. Let $d = p_1^{\beta_1}\cdots p_k^{\beta_k}$ be a proper divisor of n and $x \in A_d$. Then, in $\Gamma(\mathbb{Z}_n)$, $e(x) = 2$ if and only if either $\beta_i \ne 0$ for all $i = 1,\dots,k$ or $d = p_1\cdots p_{\ell-1}\cdot p_{\ell+1}\cdots p_w\cdot p_{w+1}^{\alpha_{w+1}}\cdots p_k^{\alpha_k}$ for some $\ell \in \{1,\dots,w\}$.*

Proof. Since $d = p_1^{\beta_1}\cdots p_k^{\beta_k}$ is a proper divisor of n, we have $\beta_i \in \{0,1\}$ for all $i = 1,\dots,w$ and $\beta_i \in \{0,1,\dots,\alpha_i\}$ for all $i = w+1,\dots,k$ together with $d \ne 1$ and $d \ne n$.

Let $x \in A_d$ and $y \in Z(\mathbb{Z}_n)^* \setminus A_{\langle\frac{n}{d}\rangle}$. Implies that $d(x,y) \ne 1$.

($\Leftarrow$): Assume that $\beta_i \ne 0$ for all $i = 1,\dots,k$. Since $d(x,y) \ne 1$, we have y of the form $p_1^{\gamma_1}\cdots p_k^{\gamma_k}$ with $\gamma_r < \alpha_r - \beta_r$ for some $r \in \{1,\dots,k\}$. Choose appropriate $z = p_1^{\lambda_1}\cdots p_k^{\lambda_k}$ where $\lambda_i = \alpha_i$ if $\gamma_i = 0$ and $\lambda_i = \alpha_i - 1$ if $\gamma_i \ge 1$. Clearly z is adjacent to y. Since $\beta_i \ge 1$ for all $i = 1,\dots,k$, we have $z \in A_{\langle\frac{n}{d}\rangle}$. Therefore $x - z - y$ is a path in $\Gamma(\mathbb{Z}_n)$ and thus $d(x,y) = 2$.

Assume that $w \neq 0$ and $\beta_\ell = 0$ for unique $\ell \in \{1, \ldots, w\}$ together with $\beta_i = \alpha_i$ for all $i \in \{1, \ldots, \ell-1, \ell+1, \ldots, k\}$. In this case, y is of the form $p_1^{\gamma_1} \cdots p_{\ell-1}^{\gamma_{\ell-1}} \cdot p_{\ell+1}^{\gamma_{\ell+1}} \cdots p_k^{\gamma_k}$ where $0 \leq \gamma_i \leq \alpha_i$ for $i \in \{1, \ldots, \ell-1, \ell+1, \ldots, k\}$. Now, select $z = p_1^{\alpha_1-\gamma_1} \cdots p_{\ell-1}^{\alpha_{\ell-1}-\gamma_{\ell-1}} \cdot p_\ell \cdot p_{\ell+1}^{\alpha_{\ell+1}-\gamma_{\ell+1}} \cdots p_k^{\alpha_k-\gamma_k} \in Z(\mathbb{Z}_n)^*$. Therefore, $x - z - y$ is a path in $\Gamma(\mathbb{Z}_n)$ and so $d(x, y) = 2$.

($\Rightarrow$): Let $e(x) = 2$. Assume that $\beta_t = 0$ for some $t \in \{1, \ldots, k\}$.

If $w = 0$, then $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ where $\alpha_i \geq 2$ for all $i = 1, \ldots, k$. Let us choose $y = p_t^{\alpha_t} \in Z(\mathbb{Z}_n)^*$. Clearly, in $\Gamma(\mathbb{Z}_n)$, the vertex x is adjacent to the elements in $Z(\mathbb{Z}_n)^*$ of the form $z = p_1^{\lambda_1} \cdots p_{t-1}^{\lambda_{t-1}} \cdot p_t^{\alpha_t} \cdot p_{t+1}^{\lambda_{t+1}} \cdots p_k^{\lambda_k}$ where $\alpha_i - \beta_i \leq \lambda_i \leq \alpha_i$. Since the vertices of $\Gamma(\mathbb{Z}_n)$ are the proper divisors of n, we have $\lambda_s \neq \alpha_s$ for some $s \in \{1, \ldots, t-1, t+1, \ldots, k\}$. Implies that y is not adjacent to any such z so that $d(x, y) \neq 2$. Thus $w \neq 0$.

We claim that there exist a unique $\ell \in \{1, \ldots, w\}$ such that $\beta_\ell = 0$ together with $\beta_i = \alpha_i$ for all $i \in \{1, \ldots, \ell - 1, \ell+1, \ldots, k\}$. Suppose, on the contrary, that there exist $\ell, \ell' \in \{1, \ldots, w\}$ such that $\beta_\ell = \beta_{\ell'} = 0$. Without loss of generality, let us take $\ell < \ell'$. Select $y = p_\ell \in Z(\mathbb{Z}_n)^* \setminus A_{\langle \frac{n}{d} \rangle}$. Note that every vertex $z \in A_{\langle \frac{n}{d} \rangle}$ is of the form $z = p_1^{\gamma_1} \cdots p_{\ell-1}^{\gamma_{\ell-1}} \cdot p_\ell^{\alpha_\ell} \cdot p_{\ell+1}^{\gamma_{\ell+1}} \cdots p_{\ell'-1}^{\gamma_{\ell'-1}} \cdot p_{\ell'}^{\alpha_{\ell'}} \cdot p_{\ell'+1}^{\gamma_{\ell'+1}} \cdots p_k^{\gamma_k}$ where $\alpha_i - \beta_i \leq \gamma_i \leq \alpha_i$. By Remark 2.3, in $\Gamma(\mathbb{Z}_n)$, the neighborhood of each vertex in $A_{\langle \frac{n}{d} \rangle}$ is $A_{\langle d \rangle}$. Since $\beta_\ell = 0$, we get $y \notin A_{\langle d \rangle}$. Therefore y is not adjacent to any of the elements in $A_{\langle \frac{n}{d} \rangle}$ and so $d(x, y) > 2$, a contradiction. Thus $\beta_\ell = 0$ for unique $\ell \in \{1, \ldots, w\}$. Next we have to prove that $\beta_i = \alpha_i$ for all $i \in \{1, \ldots, \ell-1, \ell+1, \ldots, k\}$. If not, $\beta_s < \alpha_s$ for some $s \in \{1, \ldots, \ell-1, \ell+1, \ldots, k\}$. Let us take $y = p_\ell \in Z(\mathbb{Z}_n)^* \setminus A_{\langle \frac{n}{d} \rangle}$. Note that any element in $A_{\langle \frac{n}{d} \rangle}$ is of the form $p_1^{\gamma_1} \cdots p_{\ell-1}^{\gamma_{\ell-1}} \cdot p_\ell \cdot p_{\ell+1}^{\gamma_{\ell+1}} \cdots p_k^{\gamma_k}$ where $\alpha_i - \beta_i \leq \gamma_i \leq \alpha_i$ and $\gamma_{s'} < \alpha_{s'}$ for some $s' \in \{1, \ldots, \ell-1, \ell+1, \ldots, k\}$. Implies that y is not adjacent to any of the elements in $A_{\langle \frac{n}{d} \rangle}$ and so $d(x, y) > 2$, a contradiction. Thus, the claim holds true. □

We are now in a position to state and prove the third theorem which determines the formula for $W(\Gamma(\mathbb{Z}_n))$ for all the remaining cases of n.

Theorem 3.8. *Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where p_i is a prime, $\alpha_i \in \mathbb{N}$ with at least one $\alpha_i \geq 2$ and $2 \leq k \in \mathbb{N}$. Rearrange p_i's such that $n = p_1 \cdots p_w \cdot p_{w+1}^{\alpha_{w+1}} \cdots p_k^{\alpha_k}$ where $\alpha_i \geq 2$ for all $i = w+1, \ldots, k$. (In case of $\alpha_i \geq 2$ for all $i = 1, \ldots, k$, take $w = 0$). Let $d_j = p_1^{\beta_1} \cdots p_k^{\beta_k}$ be a proper divisor of n for all $j = 1, \ldots, \left(\prod_{i=1}^k (\alpha_i + 1) - 2\right)$. Arrange d_j's in such a way that*

- *for $1 \leq j \leq w$, let $d_j = p_1 \cdots p_{j-1} \cdot p_{j+1} \cdots p_w \cdot p_{w+1}^{\alpha_{w+1}} \cdots p_k^{\alpha_k}$;*
- *for $w+1 \leq j \leq w + \prod_{i=1}^k \left(\left\lceil \frac{\alpha_i+1}{2} \right\rceil \right) - 1$, let $\beta_i \geq \lceil \alpha_i/2 \rceil$ for all $i = 1, \ldots, k$ and, for $w + \prod_{i=1}^k \left\lceil \frac{\alpha_i+1}{2} \right\rceil \leq j \leq w + \prod_{i=1}^k \alpha_i - 1$, let $\beta_i \geq 1$ and $\beta_{i'} < \lceil \alpha_{i'}/2 \rceil$ for all $i = 1, \ldots, k$ and some $i' \in \{1, \ldots, k\}$;*
- *for the remaining d_j's, notate $j = w + \prod_{i=1}^k \alpha_i, \ldots, \prod_{i=1}^k (\alpha_i + 1) - 2$. In this case, let $Z_j = \{i \in \{1, \ldots, k\} : \beta_i = 0\}$ and let $Z_j = \{i_1, \ldots, i_{r(j)}\}$. For $1 \leq \ell \leq \sigma_j$, define $\tau_{\ell(j)} = p_{i_1}^{\gamma_{i_1}} \cdots p_{i_{r(j)}}^{\gamma_{i_{r(j)}}}$ where $0 \leq \gamma_{i_s} \leq \alpha_{i_s}$ for all $1 \leq s \leq r(j)$. In addition, if $\beta_m = \alpha_m$ for all $m \in \{1, \ldots, k\} \setminus Z_j$, then there exists $t \in \{1, \ldots, r(j)\}$ such that $\gamma_{i_t} < \alpha_{i_t}$.*

Then

$$W(\Gamma(\mathbb{Z}_n)) = \frac{1}{2}\left[\sum_{j=1}^{\left(\prod_{i=1}^k(\alpha_i+1)\right)-2} \left(\phi(\frac{n}{d_j}) \cdot \left(2(n - \phi(n)) - 3 - d_j\right)\right)\right] + \frac{1}{2}\left[\sum_{j=w+\prod_{i=1}^k \alpha_i}^{\left(\prod_{i=1}^k(\alpha_i+1)\right)-2} \left(\phi(\frac{n}{d_j}) \cdot \left(\sum_{\ell(j)=1}^{\sigma_j} \phi(\frac{n}{\tau_{\ell(j)}})\right)\right)\right] + \frac{1}{2}\left(\sum_{j=w+1}^{\left(\prod_{i=1}^k \left\lceil \frac{\alpha_i+1}{2} \right\rceil\right)-1} \phi(\frac{n}{d_j})\right).$$

where

$$\sigma_j = \begin{cases} \left(\prod_{s=1}^{r(j)} (\alpha_{i_s} + 1)\right) - 2 & \text{if } \beta_m = \alpha_m \text{ for all } m \in \{1, \ldots, k\} \setminus Z_j \\ \left(\prod_{s=1}^{r(j)} (\alpha_{i_s} + 1)\right) - 1 & \text{otherwise.} \end{cases}$$

Proof. Let $n = p_1 \cdots p_w \cdot p_{w+1}^{\alpha_{w+1}} \cdots p_k^{\alpha_k}$. In case of $\alpha_i \geq 2$ for all $i = 1, \ldots, k$, take $w = 0$. Here the number of proper divisors of n is $\prod_{i=1}^k (\alpha_i + 1) - 2$. For $j \in \{1, \ldots, \prod_{i=1}^k (\alpha_i + 1) - 2\}$, choose an arbitrary divisor $d_j = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdots p_k^{\beta_k}$ where $0 \leq \beta_i \leq \alpha_i$ for all $i = 1, \ldots, k$. Let $x \in A_{d_j}$. Then, by Proposition 2.4,

$$|N_1(x)| = deg(x) = \begin{cases} d_j - 2 & \text{if } \beta_i \geq \lceil \alpha_i/2 \rceil \text{ for all } i = 1, \ldots, k \\ d_j - 1 & \text{otherwise.} \end{cases} \tag{6}$$

Lemma 3.7 implies that

- We claim that $e(x) = 2$ if and only if either $\beta_i \neq 0$ for all $i = 1, \ldots, k$ or, in case of $w \neq 0$, there exists a unique $r \in \{1, \ldots, w\}$ such that $\beta_r = 0$ together with $\beta_s = \alpha_s$ for all $s \in \{w+1, \ldots, k\}$.
- $e(x) = 3$ if and only if (i) $\beta_r \neq 0$ for all $r \in \{1, \ldots, w\}$ and $\beta_s = 0$ for some $s \in \{w+1, \ldots, k\}$ or (ii) there exists a unique $r' \in \{1, \ldots, w\}$ such that $\beta_{r'} = 0$ together with $\beta_s < \alpha_s$ for some $s \in \{w+1, \ldots, k\}$ or (iii) $\beta_{r'} = \beta_{r''} = 0$ for some $r', r'' \in \{1, \ldots, w\}$.
- $e(x) = e(y)$ for all $x, y \in A_{d_j}$ and for any $j \in \{1, \ldots, \prod_{i=1}^{k}(\alpha_i+1) - 2\}$.

Case 1: Assume that $e(x) = 2$ for $x \in A_{d_j}$. Then $|N_2(x)| = |Z(\mathbb{Z}_n)^* \setminus \{x\}| - deg(x)$. So, by Eq. (6), we get

$$|N_2(x)| = \begin{cases} n - \phi(n) - d_j & \text{if } \beta_i \geq \left\lceil \frac{\alpha_i}{2} \right\rceil \text{ for all } 1 \leq i \leq k \\ n - \phi(n) - d_j - 1 & \text{otherwise} \end{cases}$$

Thus

$$\begin{aligned} d(x|G) &= \begin{cases} (d_j - 2) + 2(n - \phi(n) - d_j) & \text{if } \beta_i \geq \left\lceil \frac{\alpha_i}{2} \right\rceil \text{ for all } 1 \leq i \leq k \\ (d_j - 1) + 2(n - \phi(n) - d_j - 1) & \text{otherwise} \end{cases} \\ &= \begin{cases} 2(n - \phi(n)) - d_j - 2 & \text{if } \beta_i \geq \left\lceil \frac{\alpha_i}{2} \right\rceil \text{ for all } 1 \leq i \leq k \\ 2(n - \phi(n)) - d_j - 3 & \text{otherwise.} \end{cases} \end{aligned} \tag{7}$$

Case 2: Assume that $e(x) = 3$ for $x \in A_{d_j}$ where $d_j = p_1^{\beta_1} \cdots p_k^{\beta_k}$. Implies that $\beta_i \in \{0, 1\}$ for $i = 1, \ldots, w$. Now, rearrange p_i's such that $\beta_i \neq 0$ for $1 \leq i \leq w'$ and $w+1 \leq i \leq k'$ and, $\beta_i = 0$ for $w'+1 \leq i \leq w$ and $k'+1 \leq i \leq k$. Clearly, these rearrangements of p_i's does not affect the value of w. Therefore $d_j = p_1 \cdots p_{w'} \cdot p_{w+1}^{\beta_{w+1}} \cdots p_{k'}^{\beta_{k'}}$. Here there are three possibilities (i) $w' = w$ and $k' < k$ (ii) $w' = w - 1$, $k' \leq k$ and $\beta_s < \alpha_s$ for some $s \in \{w+1, \ldots, k\}$ (iii) $w' \leq w - 2$ and $k' \leq k$.

Let $y \in A_{d_{j'}}$ and $d_{j'} = p_1^{\gamma_1} \cdots p_k^{\gamma_k}$ where $0 \leq \gamma_i \leq \alpha_i$ for all $i = 1, 2, \ldots, k$.

Claim. *$d(x, y) = 2$ if and only if $\gamma_\ell < \alpha_\ell - \beta_\ell$ for some $\ell \in \{1, \ldots, k\}$ and $\gamma_m \neq 0$ for at least one $m \in \{1, 2, \ldots, w', w+1, \ldots, k'\}$.*

($\Rightarrow$): Assume that $d(x, y) = 2$. Suppose, on the contrary, that either $\gamma_i \geq \alpha_i - \beta_i$ for all $i = 1, \ldots, k$ or $d_{j'} = p_{w'+1}^{\gamma_{w'+1}} \cdots p_w^{\gamma_w} \cdot p_{k'+1}^{\gamma_{k'+1}} \cdots p_k^{\gamma_k}$ where $0 \leq \gamma_i \leq \alpha_i$ for all $i = w'+1, \ldots, w, k'+1, \ldots, k$.

If $d_{j'} = p_1^{\gamma_1} \cdots p_k^{\gamma_k}$ where $\gamma_i \geq \alpha_i - \beta_i$ for all $i = 1, \ldots, k$, then $d(x, y) = 1$, a contradiction.

Let $d_{j'} = p_{w'+1}^{\gamma_{w'+1}} \cdots p_w^{\gamma_w} \cdot p_{k'+1}^{\gamma_{k'+1}} \cdots p_k^{\gamma_k}$ where $0 \leq \gamma_i \leq \alpha_i$. If $\beta_i = \alpha_i$ for all $i = w+1, \ldots, k'$ and $\gamma_{i'} = \alpha_{i'}$ for all $i' = w'+1, \ldots, w, k'+1, \ldots, k$, then $d(x, y) = 1$, a contradiction. Therefore, assume that either $\beta_s < \alpha_s$ for some $s \in \{w+1, \ldots, k'\}$ or $\gamma_t < \alpha_t$ for some $t \in \{w'+1, \ldots, w, k'+1, \ldots, k\}$. Implies that $d(x, y) \neq 1$. In both cases, y is adjacent to all the elements in $Z(\mathbb{Z}_n)^*$ of the form $z = p_1 \cdots p_{w'} \cdot p_{w'+1}^{\lambda_{w'+1}} \cdots p_w^{\lambda_w} \cdot p_{w+1}^{\alpha_{w+1}} \cdots p_{k'}^{\alpha_{k'}} \cdot p_{k'+1}^{\lambda_{k'+1}} \cdots p_k^{\lambda_k}$ with $\alpha_i - \gamma_i \leq \lambda_i \leq \alpha_i$ for all $i = w'+1, \ldots, w, k'+1, \ldots, k$. Since z is a proper divisor of n, there exist $t' \in \{w'+1, \ldots, w, k'+1, \ldots, k\}$ such that $\lambda_{t'} \neq \alpha_{t'}$. The corresponding $\beta_{t'} = 0$ so that z is not adjacent to x and so $d(x, y) > 2$, a contradiction.

($\Leftarrow$): Assume that $\gamma_\ell < \alpha_\ell - \beta_\ell$ for some $\ell \in \{1, \ldots, k\}$ and $\gamma_m \neq 0$ for some $m \in \{1, 2, \ldots, w', w+1, \ldots, k'\}$. Since $\gamma_\ell < \alpha_\ell - \beta_\ell$, we get $d(x, y) \neq 1$. Clearly

$$z = p_1^{\alpha_1} \cdots p_{m-1}^{\alpha_{m-1}} \cdot p_m^{\alpha_m - 1} \cdot p_{m+1}^{\alpha_{m+1}} \cdots p_k^{\alpha_k} \in Z(\mathbb{Z}_n)^*.$$

Since $\gamma_m \geq 1$ and $\beta_m \geq 1$, we have $x - z - y$ as a path in $\Gamma(\mathbb{Z}_n)$ so that $d(x, y) = 2$. Thus, the claim holds true.

For $x \in A_{d_j}$ and $1 \leq m \leq 3$, let us denote σ_{jm} as the number of proper divisors d of n such that $d(x, y) = m$ for $y \in A_d$. So, to find $|N_2(x)|$ and $|N_3(x)|$, we have to calculate σ_{j2} and σ_{j3}. Note that, in this case, $d_j = p_1 \cdots p_{w'} \cdot p_{w+1}^{\beta_{w+1}} \cdots p_{k'}^{\beta_{k'}}$ where either $w' \leq w - 1$ or $k' \leq k - 1$. Clearly, x is adjacent to all the vertices of the sets A_d with d of the form $p_1^{\gamma_1} \cdots p_k^{\gamma_k}$ where $\gamma_i \geq \alpha_i - \beta_i$. So $\sigma_{j1} = \left(\prod_{i=1}^{k}(\beta_i + 1)\right) - 1$. Also, by the claim, the sets A_d's of distance three from x are of the form $p_{w'+1}^{\gamma_{w'+1}} \cdots p_w^{\gamma_w} \cdot p_{k'+1}^{\gamma'_k+1} \cdots p_k^{\gamma_k}$ where $0 \leq \gamma_i \leq \alpha_i$. The point to note is, if $\beta_\ell = \alpha_\ell$ for all $\ell \in \{w+1, \ldots, k'\}$ and $d = p_{w'+1}^{\alpha_{w'+1}} \cdots p_w^{\alpha_w} \cdot p_{k'+1}^{\alpha'_k+1} \cdots p_k^{\alpha_k}$, then $d(x, y) = 1$. Therefore,

$$\sigma_{j3} = \begin{cases} \left(\prod_{i=w'+1}^{w}(\alpha_i + 1)\right) \cdot \left(\prod_{i=k'+1}^{k}(\alpha_i + 1)\right) - 2 & \text{if } \beta_\ell = \alpha_\ell \ \forall\ \ell \in \{w+1, \ldots, k'\} \\ \left(\prod_{i=w'+1}^{w}(\alpha_i + 1)\right) \cdot \left(\prod_{i=k'+1}^{k}(\alpha_i + 1)\right) - 1 & \text{otherwise.} \end{cases}$$

Thus, we take $\sigma_{j2} = \left(\prod_{i=1}^{k}(\alpha_i + 1) - 2\right) - \sigma_{j1} - \sigma_{j3}$.

In this case $d_j = p_1 \cdots p_{w'} \cdot p_{w+1}^{\beta_{w+1}} \cdots p_{k'}^{\beta_{k'}}$ where (i) $w' = w$ and $k' < k$ or (ii) $w' = w - 1$, $k' \leq k$ and $\beta_s < \alpha_s$ for some $s \in \{w+1, \ldots, k\}$ or (iii) $w' \leq w - 2$ and $k' \leq k$. So the sets A_{d_m} corresponding to σ_{j3} is $d_m = p_{w'+1}^{\gamma_{w'+1}} \cdots p_w^{\gamma_w} \cdot p_{k'+1}^{\gamma_{k'+1}} \cdots p_k^{\gamma_k}$

with either $\beta_\ell < \alpha_\ell$ for some $\ell \in \{w+1, \ldots, k'\}$ or $\gamma_t < \alpha_t$ for some $t \in \{w'+1, \ldots, w, k'+1, \ldots, k\}$. Let us denote these d_m's by $\tau_1, \ldots, \tau_{\sigma_{j3}}$. So

$$|N_3(x)| = \sum_{s=1}^{\sigma_{j3}} \phi(\frac{n}{\tau_s}), \tag{8}$$

and so

$$|N_2(x)| = n - \phi(n) - d_j - \sum_{s=1}^{\sigma_{j3}} \phi(\frac{n}{\tau_s}) - 1. \tag{9}$$

Therefore, by Eqs. (8) and (9), in case of $e(x) = 3$, we have

$$\begin{aligned} d(x|G) &= (d_j - 1) + 3\left(\sum_{s=1}^{\sigma_{j3}} \phi(\frac{n}{\tau_s})\right) + 2\left(n - \phi(n) - d_j - \sum_{s=1}^{\sigma_{j3}} \phi(\frac{n}{\tau_s}) - 1\right) \\ &= 2\,(n - \phi(n)) - d_j + \sum_{s=1}^{\sigma_{j3}} \phi(\frac{n}{\tau_s}) - 3. \end{aligned} \tag{10}$$

Now, to compute the Wiener index, it is required to find the number of choices for j's such that $e(x) = 2$ for $x \in A_{d_j}$. Clearly, by Lemma 3.7, the number of sets A_{d_j} in $\mathbb{Z}_n$ with $e(x) = 2$ for any $x \in A_{d_j}$ is $w + \prod_{i=1}^{k} \alpha_i - 1$. Note that, by Eq. (7), there are two possibilities available in case of $e(x) = 2$. In this case, the number of choices for d_j with $\beta_i \geq \lceil \frac{\alpha_i}{2} \rceil$ for all $1 \leq i \leq k$ is $\left(\prod_{i=1}^{k} \lceil \frac{\alpha_i+1}{2} \rceil\right) - 1$.

For $1 \leq j \leq w$, let $d_j = p_1 \cdots p_{j-1} \cdot p_{j+1} \cdots p_w \cdot p_{w+1}^{\alpha_{w+1}} \cdots p_k^{\alpha_k}$ and, for $1 \leq j \leq \prod_{i=1}^{k} \alpha_i - 1$, let $d_j = p_1^{\beta_1} \cdots p_k^{\beta_k}$ where $\beta_i \neq 0$ for all $i = 1, \ldots, k$.

Thus, by Eqs. (7) and (10),

$$\begin{aligned} W(\Gamma(\mathbb{Z}_n)) &= \frac{1}{2}\left[\sum_{j=1}^{w+\prod_{i=1}^{k}\alpha_i - 1} \left(\phi(\frac{n}{d_j}) \cdot d(x|G)_{e(x)=2}\right)\right] \\ &\quad + \frac{1}{2}\left[\sum_{j=w+\prod_{i=1}^{k}\alpha_i}^{\left(\prod_{i=1}^{k}(\alpha_i+1)\right)-2} \left(\phi(\frac{n}{d_j}) \cdot d(x|G)_{e(x)=3}\right)\right] \\ &= \frac{1}{2}\left[\sum_{j=1}^{w+\prod_{i=1}^{k}\alpha_i - 1} \left(\phi(\frac{n}{d_j}) \cdot \left(2(n - \phi(n)) - 3 - d_j\right)\right)\right] + \frac{1}{2}\left(\sum_{j=w+1}^{\left(\prod_{i=1}^{k}\lceil\frac{\alpha_i+1}{2}\rceil\right)-1} \phi(\frac{n}{d_j})\right) \\ &\quad + \frac{1}{2}\left[\sum_{j=w+\prod_{i=1}^{k}\alpha_i}^{\left(\prod_{i=1}^{k}(\alpha_i+1)\right)-2} \left(\phi(\frac{n}{d_j}) \cdot \left(2\,(n - \phi(n)) - d_j - 3 + \sum_{s=1}^{\sigma_{j3}} \phi(\frac{n}{\tau_s})\right)\right)\right] \\ &= \frac{1}{2}\left[\sum_{j=1}^{\left(\prod_{i=1}^{k}(\alpha_i+1)\right)-2} \left(\phi(\frac{n}{d_j}) \cdot \left(2(n - \phi(n)) - d_j - 3\right)\right)\right] \\ &\quad + \frac{1}{2}\left[\sum_{j=w+\prod_{i=1}^{k}\alpha_i}^{\left(\prod_{i=1}^{k}(\alpha_i+1)\right)-2} \left(\phi(\frac{n}{d_j}) \cdot \left(\sum_{s=1}^{\sigma_{j3}} \phi(\frac{n}{\tau_s})\right)\right)\right] + \frac{1}{2}\left(\sum_{j=w+1}^{\left(\prod_{i=1}^{k}\lceil\frac{\alpha_i+1}{2}\rceil\right)-1} \phi(\frac{n}{d_j})\right). \quad \square \end{aligned}$$

Reddy et al. [22] have found $W(\Gamma(\mathbb{Z}_n))$ for $n = p^2 q$. But there is a flaw in the proof of Theorem 5.2 in [22]. More specifically, the authors of [22] missed out the distance 3 cases. For instance, if $x \in A_p$ and $y \in A_q$, then x is adjacent to only the vertices of A_{pq} but y is not adjacent to any of the vertex of A_{pq}. Therefore $d(x, y) = 3$. The following result, gives the exact value of $W(\Gamma(\mathbb{Z}_n))$ for $n = p^2 q$.

Corollary 3.9. *Let p_1 and p_2 be distinct primes. Then*

$$W(\Gamma(\mathbb{Z}_{p_1 p_2^2})) = \frac{1}{2}\left[6p_1p_2^3 + 2p_1^2p_2^2 - 12p_1p_2^2 + 2p_2^4 - 6p_2^3 + 3p_2^2 + 3p_2 + 2\right].$$

Let us conclude this paper with an illustration for Theorem 3.8.

Remark 3.10. Consider $n = 5\cdot 7\cdot 2^3\cdot 3^2 = 2520$. Here $w = 2$, $\prod_{i=1}^{k}\left(\left\lceil\frac{\alpha_i+1}{2}\right\rceil\right) - 1 = 3$, $\prod_{i=1}^{k}\alpha_i - 1 = 5$ and the number of proper divisors is $\prod_{i=1}^{k}(\alpha_i+1) - 2 = 46$.

- $d_1 = 7\cdot 2^3\cdot 3^2$ and $d_2 = 5\cdot 2^3\cdot 3^2$.
- Let $d_3 = 5\cdot 7\cdot 2^3\cdot 3,\ d_4 = 5\cdot 7\cdot 2^2\cdot 3,\ d_5 = 5\cdot 7\cdot 2^2\cdot 3^2$ and $d_6 = 5\cdot 7\cdot 2\cdot 3^2,\ d_7 = 5\cdot 7\cdot 2\cdot 3$.
- Let $d_8 = 5\cdot 7\cdot 2^3,\ d_9 = 5\cdot 7\cdot 2^2,\ d_{10} = 5\cdot 7\cdot 2,\ d_{11} = 5\cdot 7\cdot 3^2,\ d_{12} = 5\cdot 7\cdot 3,\ d_{13} = 5\cdot 2\cdot 3^2,\ d_{14} = 5\cdot 2\cdot 3,\ d_{15} = 5\cdot 2^2\cdot 3^2,\ d_{16} = 5\cdot 2^2\cdot 3,\ d_{17} = 5\cdot 2^3\cdot 3,\ d_{18} = 7\cdot 2\cdot 3^2,\ d_{19} = 7\cdot 2\cdot 3,\ d_{20} = 7\cdot 2^2\cdot 3^2,\ d_{21} = 7\cdot 2^2\cdot 3,\ d_{22} = 7\cdot 2^3\cdot 3,\ d_{23} = 5\cdot 7,\ d_{24} = 5\cdot 2^3,\ d_{25} = 5\cdot 2^2,\ d_{26} = 5\cdot 2,\ d_{27} = 5\cdot 3^2,\ d_{28} = 5\cdot 3,\ d_{29} = 7\cdot 2^3,\ d_{30} = 7\cdot 2^2,\ d_{31} = 7\cdot 2,\ d_{32} = 7\cdot 3^2,\ d_{33} = 7\cdot 3,\ d_{34} = 2^3\cdot 3^2,\ d_{35} = 2^3\cdot 3,\ d_{36} = 2^2\cdot 3^2,\ d_{37} = 2^2\cdot 3,\ d_{38} = 2\cdot 3^2,\ d_{39} = 2\cdot 3,\ d_{40} = 5,\ d_{41} = 7,\ d_{42} = 2^3,\ d_{43} = 2^2,\ d_{44} = 2,\ d_{45} = 3^2$ and $d_{46} = 3$.

In general, take $d_j = 5^{\beta_1}\cdot 7^{\beta_2}\cdot 2^{\beta_3}\cdot 3^{\beta_4}$ where $\beta_i \in \{1,\dots,\alpha_i\}$ for $j = 1,\dots,46$. For instance, we elaborate the terms in the formula of $W(\Gamma(\mathbb{Z}_n))$ for d_{33} and d_{41}.

For $d_{33} = 7\cdot 3$; we have $\beta_1 = \beta_3 = 0$. Implies that $Z_{33} = \{1, 3\}$. That is $r(33) = 2$, $i_1 = 1$ and $i_2 = 3$. Since $\beta_4 < \alpha_4$, we have $\sigma_j = \left(\prod_{s=1}^{2}(\alpha_{i_s}+1)\right) - 1 = 7$. Consequently $\tau_{1(33)} = 2,\ \tau_{2(33)} = 2^2,\ \tau_{3(33)} = 2^3,\ \tau_{4(33)} = 5,\ \tau_{5(33)} = 5\cdot 2,\ \tau_{6(33)} = 5\cdot 2^2$ and $\tau_{7(33)} = 5\cdot 2^3$. Therefore $\sum_{\ell(33)=1}^{7}\phi(\frac{n}{\tau_{\ell(33)}}) = 288 + 144 + 144 + 72 + 36 + 36 + 144 = 864$.

For $d_{41} = 7$; we have $Z_{41} = \{1, 3, 4\}$. Since $\beta_2 = \alpha_2$, $\sigma_j = \left(\prod_{s=1}^{3}(\alpha_{i_s}+1)\right) - 2 = 22$. Consequently, $\tau_{1(33)} = 2,\ \tau_{2(33)} = 2^2,\ \tau_{3(33)} = 2^3,\ \tau_{4(33)} = 3,\ \tau_{5(33)} = 3^2,\ \tau_{6(33)} = 5,\ \tau_{7(33)} = 2\cdot 3,\ \tau_{8(33)} = 2^2\cdot 3,\ \tau_{9(33)} = 2^3\cdot 3,\ \tau_{10(33)} = 2\cdot 3^2,\ \tau_{11(33)} = 2^2\cdot 3^2,\ \tau_{12(33)} = 2^3\cdot 3^2,\ \tau_{13(33)} = 5\cdot 2,\ \tau_{14(33)} = 5\cdot 2^2,\ \tau_{15(33)} = 5\cdot 2^3,\ \tau_{16(33)} = 5\cdot 3,\ \tau_{17(33)} = 5\cdot 3^2,\ \tau_{18(33)} = 5\cdot 2\cdot 3,\ \tau_{19(33)} = 5\cdot 2^2\cdot 3,\ \tau_{20(33)} = 5\cdot 2^3\cdot 3,\ \tau_{21(33)} = 5\cdot 2\cdot 3^2$ and $\tau_{22(33)} = 5\cdot 2^2\cdot 3^2$. Therefore $\sum_{\ell(41)=1}^{22}\phi(\frac{n}{\tau_{\ell(11)}}) = 288+144+144+192+96+144+96+56+48+48+24+24+72+36+36+32+24+24+12+12+12+6 = 1570$.

Note that $n - \phi(n) = 2520 - 576 = 1944$. So

$$\begin{aligned} W(\Gamma(\mathbb{Z}_{2520})) &= \frac{1}{2}\left[\sum_{j=1}^{46}\left(\phi(\frac{n}{d_j})\cdot\left(2\times 1944 - 3 - d_j\right)\right)\right] \\ &\quad + \frac{1}{2}\left[\sum_{j=8}^{46}\left(\phi(\frac{n}{d_j})\cdot\left(\sum_{\ell(j)=1}^{\sigma_j}\phi(\frac{n}{\tau_{\ell(j)}})\right)\right)\right] + \frac{1}{2}\left(\sum_{j=3}^{5}\phi(\frac{n}{d_j})\right) \\ &= \frac{1}{2}\left[7533399 + 1351152 + 5\right] = 4442278. \end{aligned}$$

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