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On Wiener Index of Unit Graph Associated with a Commutative Ring

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Abstract. Let R be a ring with non-zero identity. The unit graph of R, denoted by $G(R)$, is an undirected graph with all the elements of R as vertices and where distinct vertices x, y are adjacent if and only if $x + y$ is a unit of R. In this paper, we investigate the Wiener index and hyper-Wiener index of $G(R)$ and explicitly determine their values.

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1 Introduction

The concept of *unit graph* was first introduced by Grimaldi in [9]. He defined the unit graph for $\mathbb{Z}_n$ based on the elements and units of $\mathbb{Z}_n$: the vertices of the unit graph $G(\mathbb{Z}_n)$ are all elements of $\mathbb{Z}_n$, and distinct elements x, y are adjacent if $x + y \in U(\mathbb{Z}_n)$, where $U(\mathbb{Z}_n)$ is the set of unit elements of $\mathbb{Z}_n$. Later on, Ashrafi et al. [4] generalized the unit graph $G(\mathbb{Z}_n)$ to $G(R)$ with the same definition, where R is an associative ring with non-zero identity, and they gave some characterizations of unit graphs. They also discussed some properties of $G(R)$ like connectedness, diameter, girth and planarity. Since then, a lot of research papers have been devoted to this topic; see, for example, [1], [3], [5], [6], [13], and [19]–[21].

In the mathematics literature, the first study of Wiener index started in 1976 [8]. Over the years many researchers have started working on the Wiener index of a graph. Based on this index, some other topological indices are also discovered. The distance between two vertices u, v of a graph G, denoted by $d_G(u, v)$, is the length of the shortest path in G beginning at u and ending at v. The Wiener index of a connected graph G is defined by

$$W(G) = \sum_{u,v \in V(G)} d_G(u, v).$$

The Wiener index, discovered by the chemist Harold Wiener [22] in 1947, is one of the topological indices. It takes a main role in the chemistry literature to find the distance between the molecules. First Wiener considered the path number for acyclic molecules. However, the above definition of $W(G)$ is defined by Hosoya [11]. The distance of a vertex u in a graph G is denoted by $d(u|G)$ and it is defined as

$$d(u|G) = \sum_{v \in V(G)} d(u, v).$$

Then the Wiener index of G is also defined as

$$W(G) = \frac{1}{2} \sum_{u \in V(G)} d(u|G).$$

In recent years, the Wiener index has become a more interesting topic in the field of graphs that are discovered from rings. There are numerous studies trying to compute or estimate the Wiener index of graphs associated to rings. But most of the studies are algorithm-oriented. For instance, the Wiener index of the zero-divisor graph of $\mathbb{Z}_n$, where $n = p^2, pq$, was investigated by Ahmadi and Nezhad in [2], and they also provided an algorithm to find the energy and the Wiener index of the zero-divisor graph of $\mathbb{Z}_n$. Similarly, when $n = p^2, pq$, the Wiener index of the total graph of $\mathbb{Z}_n$, denoted by $T(\Gamma(\mathbb{Z}_n))$, was given by Sheela and Prakash in [18], and they also wrote an algorithm to find $W(T(\Gamma(\mathbb{Z}_n)))$. Pranjali and Acharya discussed an algorithm to find the energy and the Wiener index of the unit graph $G(\mathbb{Z}_n)$ in [16], and by using the algorithm, they gave the value of $W(G(\mathbb{Z}_n))$ for $n = 2, 3, 5, 15, 48$. In 2014 Nikmehr et al., without using an algorithm, found the Wiener index of the total graph of $\mathbb{Z}_n$ for all n in [15]. Recently, Asir and Rabikka [7] calculated the Wiener index of the zero-divisor graph of $\mathbb{Z}_n$ for any positive integer n.

Let us recall some definitions in graph theory. Let G be a graph. A walk of length k in G is an alternating sequence of vertices and edges, $v_0, e_0, v_1, e_1, v_2, \ldots, v_{k-1}$, e_{k-1}, v_k, which begins and ends with vertices. A path of length k in G is a walk with all vertices distinct. The distance of two vertices x, y in G, denoted by $d(x, y)$, is the number of edges in a shortest path between x and y. If there is no path connecting the two vertices, then the distance is defined as infinite. The *eccentricity* of the vertex u, denoted by $e(u)$, is the maximum distance from u to any vertex in G, that is, $e(u) = \max_{v \in V(G)} d(u, v)$. The *diameter* of a graph is the maximum eccentricity of all vertices in the graph, and thus $\mathrm{diam}(G) = \max_{u \in V(G)} e(u)$. A *bipartite graph* is a graph with two independent sets in which the vertices in the same set are not adjacent.

The hyper-Wiener index of acyclic graphs was introduced by Milan Randic in 1993 [17]. Klein et al. [12] generalized Randic's definition for all connected graphs, as a generalization of the Wiener index. Then the hyper-Wiener index of a graph G is defined as

$$HW(G) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v) + \frac{1}{2} \sum_{u,v \in V(G)} d_G^2(u,v).$$

In this paper, we study the Wiener index and hyper-Wiener index of the unit graph of a ring. We completely determine the values of the Wiener index and hyper-Wiener index of the unit graph for a finite commutative ring with non-zero identity.

2 Results

We first recall two results from [4], which play a key role in this paper.

Proposition 2.1. [4, Proposition 2.4] *Let R be a finite ring. Then the following statements hold for the unit graph of R:*

(1) *If $2 \notin U(R)$, then the unit graph $G(R)$ is a $|U(R)|$-regular graph.*

(2) *If $2 \in U(R)$, then for every $x \in U(R)$ we have $\deg(x) = |U(R)| - 1$, and for every $x \in R \backslash U(R)$ we have $\deg(x) = |U(R)|$.*

Proposition 2.2. [4, Theorem 5.7] *Let R be a finite commutative ring. Then the following statements hold:*

(1) $\mathrm{diam}(G(R)) = 1$ *if and only if R is a field with* $\mathrm{Char}(R) = 2$

(2) $\mathrm{diam}(G(R)) = 2$ *if and only if one of the following cases occurs:*

  (a) *R is a field with* $\mathrm{Char}(R) \neq 2$;

  (b) *R is not a field and R cannot have $\mathbb{Z}_2$ as a quotient;*

  (c) *R is a local ring with maximal ideal $\mathfrak{m}$ such that $|R/\mathfrak{m}| = 2$ and $R \ncong \mathbb{Z}_2$.*

(3) $\mathrm{diam}(G(R)) = 3$ *if and only if R has $\mathbb{Z}_2$ as a quotient but cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient and R is not a local ring.*

(4) $\mathrm{diam}(G(R)) = \infty$ *if and only if R has $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient.*

Now, we give two lemmas.

Lemma 2.3. *Let R be a finite commutative ring and $x \in R$. Suppose that* $\mathrm{diam}(G(R)) = 2$ *and $A_k = \{y \in R \mid d(x,y) = k\}$. Then the following hold:*

(1) *If $2 \notin U(R)$, then $|A_1| = |U(R)|$ and $|A_2| = |R| - |U(R)| - 1$.*

(2) *If $2 \in U(R)$, then*

$$|A_1| = \begin{cases} |U(R)| & \text{if } x \notin U(R), \\ |U(R)| - 1 & \text{if } x \in U(R), \end{cases}$$

$$|A_2| = \begin{cases} |R| - |U(R)| - 1 & \text{if } x \notin U(R), \\ |R| - |U(R)| & \text{if } x \in U(R). \end{cases}$$

Proof. (1) By Proposition 2.1(1), we see that in this case the unit graph $G(R)$ is a $|U(R)|$-regular graph, so $|A_1| = |U(R)|$. As $\operatorname{diam}(G(R)) = 2$, the remaining vertices have distance 2 from x, and thus $|A_2| = |R| - |U(R)| - 1$.

(2) By Proposition 2.1(2), it is clear that we have $|A_1| = |U(R)|$ if $x \notin U(R)$ and $|A_1| = |U(R)| - 1$ if $x \in U(R)$. As $\operatorname{diam}(G(R)) = 2$, the remaining vertices have distance 2 from x, and the results follow. □

Lemma 2.4. *Let R be a finite commutative ring and $x \in R$. Suppose that $\operatorname{diam}(G(R)) = 3$ and define $A_k = \{y \in R \mid d(x,y) = k\}$. Then $|A_1| = |U(R)|$, $|A_2| = |R|/2 - 1$ and $|A_3| = |R|/2 - |U(R)|$.*

Proof. By Proposition 2.2(3), we know that $2 \notin U(R)$. So $G(R)$ is $|U(R)|$-regular, and thus $|A_1| = |U(R)|$.

Note that every finite commutative ring is isomorphic to the direct product of finite local rings (see [14]). So we may write $R \cong R_1 \times \cdots \times R_n$, where R_i is a local ring with maximal ideal $\mathfrak{m}_i$ for $1 \le i \le n$. By Proposition 2.2, we see that R is a non-local ring with $\mathbb{Z}_2$ as a quotient but cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient. Therefore, $n \ge 2$, and exactly one of the local rings R_i for $1 \le i \le k$ has $\mathbb{Z}_2$ as a quotient. Let us take R_1 to be a local ring with $R_1/\mathfrak{m}_1 \cong \mathbb{Z}_2$.

Let $x = (x_1, \ldots, x_n) \in R$. We claim that $A_2 = \{\mathfrak{m}_1 \times R_2 \times \cdots \times R_n\} \backslash \{x\}$ if $x_1 \in \mathfrak{m}_1$ and $A_2 = \{R_1 \backslash \mathfrak{m}_1 \times R_2 \times \cdots \times R_n\} \backslash \{x\}$ if $x_1 \notin \mathfrak{m}_1$.

First, let $x_1 \in \mathfrak{m}_1$. Suppose that $y = (y_1, \ldots, y_n) \in A_2$. Then $d(x,y) = 2$, and so there exists an arbitrary vertex $z = (z_1, \ldots, z_n)$ in $G(R)$ such that z is adjacent to both x and y. Clearly, $z \in A_1$, which implies $z_1 \notin \mathfrak{m}_1$. If $y_1 \notin \mathfrak{m}_1$, then $z_1 + y_1 \in \mathfrak{m}_1$ because $R_1/\mathfrak{m}_1 \cong \mathbb{Z}_2$, a contradiction of the fact that y and z are adjacent. Therefore, $y_1 \in \mathfrak{m}_1$ so that $y \in \{\mathfrak{m}_1 \times R_2 \times \cdots \times R_n\} \backslash \{x\}$.

Suppose that $y = (y_1, \ldots, y_n)$ with $y_1 \in \mathfrak{m}_1$. Then $d(x,y) \ne 1$. We have to prove $y \in A_2$. Choose $z_i \in R_i$ for $1 \le i \le n$ as follows:

$$z_i = \begin{cases} 1 & \text{if } x_i \in \mathfrak{m}_i, \\ 0 & \text{if } x_i \notin \mathfrak{m}_i. \end{cases}$$

Thus, we have the path $(x_1, \ldots, x_n) - (z_1, \ldots, z_n) - (y_1, \ldots, y_n)$ in $G(R)$, and so $d(x,y) = 2$. Therefore, $A_2 = \{\mathfrak{m}_1 \times R_2 \times \cdots \times R_n\} \backslash \{x\}$. Similarly, we can prove for the case $x_1 \notin \mathfrak{m}_1$, but the set A_2 is $\{R_1 \backslash \mathfrak{m}_1 \times R_2 \times \cdots \times R_n\} \backslash \{x\}$. Certainly, the cardinalities of both the sets are the same, which is nothing but $|R|/2 - 1$. Accordingly, $|A_2| = |R|/2 - 1$.

Note that all the remaining vertices fall into A_3 since $\operatorname{diam}(G(R)) = 3$. So

$$|A_3| = (|R| - 1) - |U(R)| - \Big(\frac{|R|}{2} - 1\Big),$$

that is, $|A_3| = |R|/2 - |U(R)|$. This completes the proofs. □

With the help of the above two lemmas, we can calculate the Wiener index of unit graphs. Our first main result is the following theorem.

Theorem 2.5. *Let R be a finite commutative ring. Then the following statements hold:*

(1) *If R is isomorphic to a field with* $\mathrm{Char}(R)=2$, *then*

$$W(G(R))=\frac{|R|^2-|R|}{2}.$$

(2) *Suppose that R is isomorphic to one of the following rings:*
   (a) *R is a field with* $\mathrm{Char}(R)\neq 2$;
   (b) *R is not a field and R cannot have $\mathbb{Z}_2$ as a quotient;*
   (c) *R is a local ring with maximal ideal $\mathfrak{m}$ such that $|R/\mathfrak{m}|=2$ and $R\not\cong\mathbb{Z}_2$.*

Then

$$W(G(R))=\begin{cases} |R|^2-\Big(\dfrac{|U(R)|}{2}+1\Big)|R| & \text{if } 2\notin U(R),\\ |R|^2-\Big(\dfrac{|U(R)|}{2}+1\Big)|R|+\dfrac{|U(R)|}{2} & \text{if } 2\in U(R).\end{cases}$$

(3) *If R is isomorphic to a non-local ring with $\mathbb{Z}_2$ as a quotient and cannot have $\mathbb{Z}_2\times\mathbb{Z}_2$ as a quotient, then*

$$W(G(R))=\frac{5}{4}|R|^2-(|U(R)|+1)|R|.$$

Proof. (1) By Proposition 2.2, we know that $\mathrm{diam}(G(R))=1$. So $G(R)$ is a complete graph, and hence the Wiener index of $G(R)$ is $(|R|^2-|R|)/2$ (see [10]).

(2) By Proposition 2.2, in this case we have $\mathrm{diam}(G(R))=2$. Let x be an arbitrary vertex of $G(R)$. Then $d(x|G)=1\cdot|A_1|+2\cdot|A_2|$.

Suppose that $2\notin U(R)$. By Lemma 2.3, $d(x|G)=|U(R)|+2(|R|-|U(R)|-1)$. Thus,

$$W(G(R))=\frac{|R|}{2}\big(|U(R)|+2(|R|-|U(R)|-1)\big)=|R|^2-\Big(\frac{|U(R)|}{2}+1\Big)|R|.$$

Suppose that $2\in U(R)$. Then by Lemma 2.3(2), we have two cases for discussion. If $x\notin U(R)$, then $d(x|G)=|U(R)|+2(|R|-|U(R)|-1)$. If $x\in U(R)$, then $d(x|G)=(|U(R)|-1)+2(|R|-|U(R)|)=2|R|-|U(R)|-1$. Thus,

$$\begin{aligned} W(G(R)) &= \frac{1}{2}\Big(\sum_{x\notin U(R)} d(x|G)+\sum_{x\in U(R)} d(x|G)\Big)\\ &= \frac{1}{2}\big((|R|-|U(R)|)(2|R|-|U(R)|-2)+|U(R)|(2|R|-|U(R)|-1)\big)\\ &= |R|^2-\Big(\frac{|U(R)|}{2}+1\Big)|R|+\frac{|U(R)|}{2}.\end{aligned}$$

(3) Again by Proposition 2.2, we have $\mathrm{diam}(G(R))=3$. Let x be an arbitrary vertex of $G(R)$. Then $d(x|G)=1\cdot|A_1|+2\cdot|A_2|+3\cdot|A_3|$. By Lemma 2.4, we get $d(x|G)=1\cdot|U(R)|+2\cdot(|R|/2-1)+3\cdot(|R|/2-|U(R)|)$. Thus,

$$\begin{aligned} W(G(R)) &= \frac{|R|}{2}\Big(|U(R)|+2\Big(\frac{|R|}{2}-1\Big)+3\Big(\frac{|R|}{2}-|U(R)|\Big)\Big)\\ &= \frac{5}{4}|R|^2-(|U(R)|+1)|R|.\end{aligned}$$

This completes the proof. □

Normally, the Wiener index is meaningless for a disconnected graph. However, we can define the Wiener index of a disconnected graph as the sum of the Wiener indices of all connected components of G. That is, $W(G) = \sum_i W(G_i)$, where G_i's are the connected components of G. Note that a *component* or *block* is a maximal connected subgraph of G.

Theorem 2.6. *If R has $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient with $R \cong R_1 \times \cdots \times R_p \times R_{p+1} \times \cdots \times R_n$, where each R_i is a local ring with maximal ideal $\mathfrak{m}_i$ for $1 \le i \le n$ and $R_j/\mathfrak{m}_j \cong \mathbb{Z}_2$ for $j = 1, \ldots, p$ with $p \ge 2$, then*

$$W(G(R)) = \begin{cases} \dfrac{3}{2^{p+1}}|R|^2 - |R| & \text{if } n = p, \\ \dfrac{5}{2^{p+1}}|R|^2 - (|U(R)| + 1)|R| & \text{otherwise.} \end{cases}$$

Proof. Let R have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient. Then by Proposition 2.2, we observe that $\operatorname{diam}(G(R)) = \infty$. We may write $R \cong R_1 \times \cdots \times R_p \times R_{p+1} \times \cdots \times R_n$, where each R_i is local with maximal ideal $\mathfrak{m}_i$ for $1 \le i \le n$ and $R_j/\mathfrak{m}_j \cong \mathbb{Z}_2$ for $j = 1, \ldots, p$ with $p \ge 2$. Let $G(R) = \bigcup_{q=1}^{m} G_q$, where G_q's are the connected components of $G(R)$. Let $S = S_1 \times \cdots \times S_p \times R_{p+1} \times \cdots \times R_n$, where $S_j = \mathfrak{m}_j$ or $R_j \backslash \mathfrak{m}_i$ for $1 \le j \le p$. Then we choose $T = T_1 \times \cdots \times T_p \times R_{p+1} \times \cdots \times R_n$ according to S as follows:

$$T_j = \begin{cases} \mathfrak{m}_j & \text{if } S_j = R_j \backslash \mathfrak{m}_j, \\ R \backslash \mathfrak{m}_j & \text{if } S_j = \mathfrak{m}_j \end{cases} \qquad \text{for } 1 \le j \le p.$$

Now our claim is to prove that the subgraph induced by the set $S \cup T$ in $G(R)$ is bipartite, $\operatorname{diam}(\langle S \cup T \rangle) \le 3$, and $\langle S \cup T \rangle = G_q$ for $q = 1, \ldots, m$. Since we have $U(R) = R_1 \backslash \mathfrak{m}_1 \times \cdots \times R_p \backslash \mathfrak{m}_p \times U(R_{p+1}) \times \cdots \times U(R_n)$, the sets S and T are independent in $G(R)$ so that $\langle S \cup T \rangle$ is bipartite. Further, $N(S) \subseteq T$ and $N(T) \subseteq S$ so that $\langle S \cup T \rangle$ is a block of $G(R)$.

Let $x = (x_1, \ldots, x_n) \in S$ with $x_i \in a_i + \mathfrak{m}_i$, $i = 1, \ldots, n$. If $y = (y_1, \ldots, y_n) \in T$ with $y_k \notin -a_k + \mathfrak{m}_k$ for all $k = p+1, \ldots, n$, then y is adjacent to x. If not, that is, $y = (y_1, \ldots, y_n) \in T$ with $y_k \in -a_k + \mathfrak{m}_k$ for some $k = p+1, \ldots, n$, then we rearrange R_k's for $p+1 \le k \le n$ such that each $y_k \notin a_k + \mathfrak{m}_k$ for $k = p+1, \ldots, p+r-1$ and each $y_k \in -a_k + \mathfrak{m}_k$ for $k = p+r, \ldots, n$, where $r \in \mathbb{N}^*$. Now we obtain

$$\begin{aligned} &(y_1, \ldots, y_n) - (x_1, \ldots, x_{p+r-1}, z_{p+r}, \ldots, z_n) \\ &- (y_1, \ldots, y_{p+r-1}, w_{p+r}, \ldots, w_n) - (x_1, \ldots, x_n), \end{aligned}$$

where $z_\ell \in b_\ell + \mathfrak{m}_\ell$ with $b_\ell + \mathfrak{m}_\ell \ne a_\ell + \mathfrak{m}_\ell, -a_\ell + \mathfrak{m}_\ell$ for $\ell = p+r, \ldots, n$ (such a $b_\ell + \mathfrak{m}_\ell$ exists because $|R_\ell/\mathfrak{m}_\ell| \ge 3$), and for $p+r \le \ell \le n$,

$$w_\ell = \begin{cases} x_\ell & \text{if } a_\ell + \mathfrak{m}_\ell \ne \mathfrak{m}_\ell, \\ z'_\ell & \text{otherwise} \end{cases}$$

with $z'_\ell \in b_\ell + \mathfrak{m}_\ell$, $z'_\ell \ne z_\ell$. Therefore, $d(x, y) = 3$. Finally, let $x \ne y \in S$. Then there exists $z = (z_1, \ldots, z_n) \in T$ such that for $p+1 \le k \le n$,

$$z_k \in \begin{cases} b_k + \mathfrak{m}_k & \text{if either } x_k, y_k \in \mathfrak{m}_k \text{ or } x_k, y_k \notin \mathfrak{m}_k, \\ a_k + \mathfrak{m}_k & \text{if either } x_k \in a_k + \mathfrak{m}_k \text{ with } a_k + \mathfrak{m}_k \neq \mathfrak{m}_k \text{ and } y_k \in \mathfrak{m}_k, \\ & \text{or } y_k \in a_k + \mathfrak{m}_k \text{ with } a_k + \mathfrak{m}_k \neq \mathfrak{m}_k \text{ and } x_k \in \mathfrak{m}_k, \end{cases}$$

where $b_k + \mathfrak{m}_k \neq a_k + \mathfrak{m}_k, -a_k + \mathfrak{m}_k$ for $k = p+1, \ldots, n$. Here $x - z - y$ is a path in $G(R)$ so that $d(x, y) = 2$. Thus, $\mathrm{diam}(\langle S \cup T \rangle) \leq 3$.

Note that there are 2^{p-1} distinct blocks in $G(R)$, where each block has $|R|/2^{p-1}$ vertices. If $n = p$, then each connected component $\langle S \cup T \rangle$ is a complete bipartite, and so the Wiener index

$$W(G(R)) = 2^{p-1}\Big(\frac{|R|}{2^P}\Big)\Big(\frac{|R|}{2^p} + 2\Big(\frac{|R|}{2^p} - 1\Big)\Big) = \frac{3}{2^{p+1}}|R|^2 - |R|.$$

If $n > p$, then

$$\begin{aligned} W(G(R)) &= 2^{p-1}\Big(\frac{|R|}{2^p}\Big)\Big(|U(R)| + 2\Big(\frac{|R|}{2^p} - 1\Big) + 3\Big(\frac{|R|}{2^p} - |U(R)|\Big)\Big) \\ &= \frac{5}{2^{p+1}}|R|^2 - (|U(R)| + 1)|R|. \end{aligned}$$

We complete the proof. $\square$

By applying the values of $|R|$ and $|U(R)|$ in Theorem 2.5, one may get the value of Wiener index of any finite commutative ring. For instance, the details are given below. For convenience, we introduce an important number-theoretic function called the Euler phi-function. Let $\phi(1) = 1$, and for any integer $n > 1$, let $\phi(n)$ denote the number of positive integers less than n and relatively prime to n. Note that $|U(\mathbb{Z}_n)| = \phi(n)$.

Example 2.7. (1) Let $R = \mathbb{Z}_n$.

If $n = 2^m$ for some $m \in \mathbb{N}^*$, then $W(G(R)) = 2^{2m} - 2^{2m-2} - 2^m$.

If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_k}$, where each p_i is an odd prime and $\alpha_i \in \mathbb{N}^*$ for $i = 1, \ldots, n$, then $W(G(R)) = n^2 - n - ((n-1)/2)\phi(n)$.

If $n = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_k}$, where each p_i is an odd prime and $\alpha_i \in \mathbb{N}^*$ for $i = 0, 1, \ldots, n$, then $W(G(R)) = 5n^2/4 - n(\phi(n) + 1)$.

(2) Let $R = \mathbb{Z}_n[x]/\langle x^2 \rangle$.

If $n = 2^m$ for some $m \in \mathbb{N}^*$, then $W(G(R)) = 2^{4m} - 2^{4m-2} - 2^{2m}$.

If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where each p_i is an odd prime and $\alpha_i \in \mathbb{N}^*$ for $i = 1, \ldots, n$, then $W(G(R)) = n^4 - n^2 - (n/2)(n^2 - 1)\phi(n)$.

If $n = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where each p_i is an odd prime and $\alpha_i \in \mathbb{N}^*$ for $i = 0, 1, \ldots, n$, then $W(G(R)) = (5/4)n^4 - (n\phi(n) + 1)n^2$.

(3) Let $R = \mathbb{Z}_n[x]/\langle x^2, xy, y^2 \rangle$.

If $n = 2^m$ for some $m \in \mathbb{N}^*$, then $W(G(R)) = 2^{6m} - 2^{6m-2} - 2^{3m}$.

If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where each p_i is an odd prime and $\alpha_i \in \mathbb{N}^*$ for $i = 1, \ldots, n$, then $W(G(R)) = n^6 - n^3 - n^2(n^3 - 1)\phi(n)$.

If $n = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where each p_i is an odd prime and $\alpha_i \in \mathbb{N}^*$ for $i = 0, 1, \ldots, n$, then $W(G(R)) = (5/4)n^6 - (n^2\phi(n) + 1)n^3$.

Applying Lemmas 2.3 and 2.4, we can also determine the value of hyper-Wiener index of unit graphs.

Theorem 2.8. *Let R be a finite commutative ring. Then the following statements hold:*

(1) *If R is isomorphic to a field with* $\mathrm{Char}(R) = 2$, *then*

$$HW(G(R)) = |R|^2 - |R|.$$

(2) *Suppose that R is isomorphic to one of the following rings:*

(a) *R is a field with* $\mathrm{Char}(R) \neq 2$;

(b) *R is not a field and R cannot have $\mathbb{Z}_2$ as a quotient;*

(c) *R is a local ring with maximal ideal $\mathfrak{m}$ such that $|R/\mathfrak{m}| = 2$ and $R \not\cong \mathbb{Z}_2$.*

Then

$$HW(G(R)) = \begin{cases} 3|R|^2 - 2|R| \cdot |U(R)| - 3|R| & \text{if } 2 \notin U(R), \\ 3|R|^2 - 2|R| \cdot |U(R)| - 3|R| + 2|U(R)| & \text{if } 2 \in U(R). \end{cases}$$

(3) *If R is isomorphic to a non-local ring with $\mathbb{Z}_2$ as a quotient and cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient, then*

$$HW(G(R)) = \frac{9}{2}|R|^2 - 5|R| \cdot |U(R)| - 3|R|.$$

Proof. (1) By Proposition 2.2, we know that $\mathrm{diam}(G(R)) = 1$. So $G(R)$ is a complete graph, and therefore the hyper-Wiener index of $G(R)$ is $|R|^2 - |R|$.

(2) By Proposition 2.2, in this case we find that $\mathrm{diam}(G(R)) = 2$. Let x be an arbitrary vertex of $G(R)$. Hence, we have $d(x|G) = 1 \cdot |A_1| + 2 \cdot |A_2|$ and $d^2(x|G) = 1 \cdot |A_1| + 4 \cdot |A_2|$.

Suppose that $2 \notin U(R)$. By Lemma 2.3, $d(x|G) = |U(R)| + 2(|R| - |U(R)| - 1)$ and $d^2(x|G) = |U(R)| + 4(|R| - |U(R)| - 1)$. Thus,

$$\begin{aligned} HW(G(R)) &= \frac{|R|}{2}\big(|U(R)| + 2(|R| - |U(R)| - 1) + |U(R)| + 4(|R| - |U(R)| - 1)\big) \\ &= 3|R|^2 - 2|R| \cdot |U(R)| - 3|R|. \end{aligned}$$

Suppose that $2 \in U(R)$. Then by Lemma 2.3(2), we have two cases for discussion.

If $x \notin U(R)$, then we observe that $d(x|G) = |U(R)| + 2(|R| - |U(R)| - 1)$ and $d^2(x|G) = |U(R)| + 4(|R| - |U(R)| - 1)$.

If $x \in U(R)$, then $d(x|G) = (|U(R)| - 1) + 2(|R| - |U(R)|) = 2|R| - |U(R)| - 1$ and $d^2(x|G) = (|U(R)| - 1) + 4(|R| - |U(R)|) = 4|R| - 3|U(R)| - 1$. Thus,

$$\begin{aligned} HW(G(R)) &= \frac{1}{2}\Big(\sum_{x \notin U(R)} d(x|G) + d^2(x|G) + \sum_{x \in U(R)} d(x|G) + d^2(x|G)\Big) \\ &= \frac{1}{2}\Big((|R| - |U(R)|)\big((2|R| - |U(R)| - 2) + (4|R| - 3|U(R)| - 4)\big) \\ &\quad + |U(R)|\big((2|R| - |U(R)| - 1) + (4|R| - 3|U(R)| - 1)\big)\Big) \\ &= 3|R|^2 - 2|R| \cdot |U(R)| - 3|R| + 2|U(R)|. \end{aligned}$$

(3) Again by Proposition 2.2, we have $\mathrm{diam}(G(R)) = 3$. Let x be an arbitrary vertex of $G(R)$. Then we can find that $d(x|G) = 1 \cdot |A_1| + 2 \cdot |A_2| + 3 \cdot |A_3|$ and $d^2(x|G) = 1 \cdot |A_1| + 4 \cdot |A_2| + 9 \cdot |A_3|$. By Lemma 2.4, we get

$$d(x|G) = 1 \cdot |U(R)| + 2 \cdot \Big(\frac{|R|}{2} - 1\Big) + 3 \cdot \Big(\frac{|R|}{2} - |U(R)|\Big),$$
$$d^2(x|G) = 1 \cdot |U(R)| + 4 \cdot \Big(\frac{|R|}{2} - 1\Big) + 9 \cdot \Big(\frac{|R|}{2} - |U(R)|\Big).$$

Thus, we obtain

$$\begin{aligned} HW(G(R)) &= \frac{|R|}{2}\Big(|U(R)| + 2\Big(\frac{|R|}{2} - 1\Big) + 3\Big(\frac{|R|}{2} - |U(R)|\Big)\Big) \\ &\quad + \frac{|R|}{2}\Big(1 \cdot |U(R)| + 4 \cdot \Big(\frac{|R|}{2} - 1\Big) + 9 \cdot \Big(\frac{|R|}{2} - |U(R)|\Big)\Big) \\ &= \frac{9}{2}|R|^2 - 5|R| \cdot |U(R)| - 3|R|. \end{aligned}$$

This completes the proof. □

We conclude the paper by writing a remark on [16].

Remark 2.9. As we mentioned in the introduction, Pranjali and Acharya [16] wrote a MATLAB program to find the Wiener index of $G(\mathbb{Z}_n)$. By using the algorithm, they obtained $W(G(\mathbb{Z}_{15})) = 52$. But note that there are eight unit elements in $\mathbb{Z}_{15}$, each of which has degree 7 in $G(\mathbb{Z}_{15})$, so that $W(G(\mathbb{Z}_{15})) \geq 56$. Now by Theorem 2.5(2), we get the value of $W(G(\mathbb{Z}_{15}))$ as 154.

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